

EVOLUTION OF A UNIVERSE FILLED WITH A CAUSAL VISCOUS FLUID

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The behaviour of solutions to the Einstein equations with a causal viscous fluid source is investigated. In this model we consider a spatially flat Robertson-Walker metric, the bulk viscosity coefficient is related to the energy density as $\zeta = \alpha \rho^m$, and the relaxation time is given by ζ/ρ . In the case $m = 1/2$ we find the exact solutions and we verify whether they satisfy the energy conditions. Besides, we study analytically the asymptotic stability of several families of solutions for arbitrary m . We find that the qualitative asymptotic behaviour in the far future is not altered by relaxation processes, but they change the behaviour in the past, introducing singular instead of deflationary evolutions or making the Universe bounce due to the violation of the energy conditions.

I. INTRODUCTION

It is believed that quantum effects played a fundamental role in the early Universe. For instance, vacuum polarisation and particle production arise from a quantum description of matter. It is known that both of them can be modelled in terms of a classical bulk viscosity [1].

Cosmological models with a viscous fluid have been studied by several authors. Some of the interesting subjects addressed by them were the effects of viscous stresses on the avoidance of the initial singularity [2], the dissipation of a primordial anisotropy [3], the production of entropy [4], and inflation and deflation [5].

Recently Pavon et al. considered a homogeneous isotropic spatially-flat universe filled with a causal viscous fluid whose bulk viscosity is related to the energy density by the power law $\zeta \sim \rho^m$ [6].

In this paper we extend and improve the analysis of this cosmological model. We present the set of equations that describe the model in section 2. In section 3 we solve them when $m = 1/2$ and give a detailed analysis of their solutions. In section 4 we study of stability of those asymptotic solutions that appear for $m \neq 1/2$ in the noncausal model by means of the Lyapunov method. The conclusions are stated in section 5.

II. THE MODEL

In the case of the homogeneous, isotropic, spatially flat Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)(dx_1^2 + dx_2^2 + dx_3^2) \quad (1)$$

only the bulk viscosity needs to be considered. Thus we replace in the Einstein equations the equilibrium pressure p by an effective pressure [4]

$$H^2 = \frac{1}{3}\rho \quad \dot{H} + 3H^2 = \frac{1}{2}(\rho - p - \sigma) \quad (2)$$

where $H = \dot{a}/a$, $\dot{} = d/dt$, ρ is the energy density, σ is the viscous pressure, and we use units $c = 8\pi G = 1$. As equation of state we take

$$p = (\gamma - 1)\rho \quad (3)$$

with a constant adiabatic index $\gamma \geq 0$, and σ has the constitutive equation of a viscoelastic fluid

$$\sigma + \tau \dot{\sigma} = -3\zeta H \quad (4)$$

Here $\zeta \geq 0$ is the bulk viscosity coefficient and τ is the bulk relaxation time and causality demands $\tau > 0$. Following [7] we choose

$$\zeta = \alpha \rho^m \quad \tau = \zeta / \rho \quad (5)$$

where α and m are constants. Using Eqs. 2–5 we get

$$\frac{\gamma}{3} \frac{|H|^r}{H_0^r} \ddot{H} + \left(\epsilon + \gamma^2 \frac{|H|^r}{H_0^r} \right) H \dot{H} + \frac{3}{2} \gamma \left(\epsilon - \frac{|H|^r}{H_0^r} \right) H^3 = 0, \quad r \neq 0, \quad \gamma \neq 0 \quad (6)$$

$$\frac{\gamma_0}{3} \ddot{H} + (\epsilon + \gamma \gamma_0) H \dot{H} + \frac{3}{2} (\epsilon \gamma - \gamma_0) H^3 = 0, \quad r = 0 \quad (7)$$

$$\ddot{H} + \frac{3^{1-r/2}}{\gamma_0} |H|^{1-r} - \frac{9}{2} H^3 = 0, \quad r \neq 0, \quad \gamma = 0 \quad (8)$$

where $\epsilon \equiv \text{sgn} H$, $r \equiv 2m - 1$, $H_0 \equiv (\gamma/\gamma_0)^{1/r}/\sqrt{3}$ and $\gamma_0 \equiv \sqrt{3}\alpha$. We assume $\alpha > 0$.

III. CASE $r = 0$

The general solution of Eq. 7 for $H > 0$ takes the following parametric form:

$$H(\eta) = \frac{\sqrt{\gamma_0/3}}{1 + \gamma \gamma_0} (A e^{\lambda_+ \eta} + B e^{\lambda_- \eta})^{1/2} \quad t(\eta) = \frac{\gamma_0/3}{1 + \gamma \gamma_0} \int \frac{d\eta}{H(\eta)} \quad (9)$$

where λ_{\pm} are the roots of $\lambda^2 + \lambda + \frac{\gamma_0(\gamma - \gamma_0)}{(1 + \gamma \gamma_0)^2} = 0$ and A, B are arbitrary integration constants.

A. One-Parameter Solutions

The one-parameter families of solutions arise when either A or B vanishes and can be obtained explicitly:

$$H_{\pm}(t) = \nu_{\pm} / \Delta t, \quad \gamma \neq \gamma_0 \quad (10)$$

$$H_+ = D, \quad H_- = \nu_0 / \Delta t, \quad \gamma = \gamma_0 \quad (11)$$

$$\nu_{\pm} = \frac{1 + \gamma \gamma_0}{3(\gamma - \gamma_0)} \left\{ 1 \pm \left[1 - \frac{4\gamma_0(\gamma - \gamma_0)}{(1 + \gamma \gamma_0)^2} \right] \right\}^{1/2} \quad \nu_0 = \frac{2}{3} \frac{\gamma_0}{1 + \gamma_0^2} \quad (12)$$

Thus, expanding solutions ‘-’ are always Friedmann, but solutions ‘+’ are Friedmann for $\gamma > \gamma_0$, de Sitter for $\gamma = \gamma_0$ and explosive for $\gamma < \gamma_0$.

B. Two-Parameter Families of Solutions

When $AB \neq 0$, $a(t)$ can be written in closed form in terms of known functions only for some values of γ and γ_0 . However, in general, we need to study the solution in the parametric form of Eq. 9. We obtain the following classification for the two-parameter families of solutions:

- A. The evolution occurs between singularities, it reaches a maximum and recollapses again. The leading behaviour near the singularities is Friedmann, as 10– for $\gamma \neq \gamma_0$ or 11– if $\gamma = \gamma_0$. These solutions have particle horizons ($0 < \nu_- < 2/3$).
- B. There is a bounce with an explosive singularity in the past and its behaviour in the future is either:
 1. asymptotically Friedmann, as 10+ for $\gamma > \gamma_0$.
 2. asymptotically de Sitter, as 11+ for $\gamma = \gamma_0$.
 3. divergent at finite times, with leading behaviour 10+ for $\gamma < \gamma_0$.
- C. The evolution begins at a singularity with a Friedmann leading behaviour, as 10– for $\gamma \neq \gamma_0$ or 11– if $\gamma = \gamma_0$; and so they have also particle horizons. Then it expands, and its behaviour in the future is like B.
- D. The evolution begins at an explosive singularity and ends at a big-crunch singularity.

C. Energy Conditions

- Dominant energy condition (DEC): $\rho \geq |p + \sigma| \Leftrightarrow -3H^2 \leq \dot{H} \leq 0$.

DEC is violated part of the time in the following families: A about the maximum (if $\nu_- \geq 1/3$); B1, about the bounce as well as for large times (if $\nu_+ > 1/3$); near the singularity in C1, C2 (if $\nu_0 < 1/3$) and C3 (if $\nu_- < 1/3$); C3, near the "explosion". DEC is violated always in families A (if $\nu_- < 1/3$), B2 and B3.

- Strong energy condition (SEC): $\rho + 3p + 3\sigma \geq 0 \Leftrightarrow \dot{H} + H^2 \leq 0$.

SEC is satisfied always in families A and C1 (if $\nu_+ \leq 1$). It is violated part of the time in the families: B1, about the bounce (if $\nu_+ \leq 1$); for large times in C1 (if $\nu_+ > 1$), C2 and C3. SEC is violated always in families B1 (if $\nu_+ > 1$), B2 and B3.

IV. CASE $r \neq 0$

We investigate the asymptotical stability of behaviors that occur in the noncausal model by means of the Lyapunov method.

A. Stability of the de Sitter solution

For the de Sitter solution $H = H_0$ we rewrite Eq. 6 as a "mechanical system".

$$\frac{d}{dt} \left[\frac{1}{2} \dot{H}^2 + V(H) \right] = -3\dot{H}^2 H \left[\gamma + \frac{1}{\gamma} \left(\frac{H_0}{H} \right)^r \right] \quad (13)$$

$$V(H) = -\frac{9}{8} H^4 + \frac{9}{2} \frac{H_0^r}{4-r} H^{4-r} \quad r \neq 4 \quad (14)$$

$$V(H) = -\frac{9}{8} H^4 + \frac{9}{2} H_0^4 \ln \frac{H}{H_0} \quad r = 4 \quad (15)$$

B. Stability of the Asymptotically Friedmann solution

For $r > 0$ it is easy to check that Eq. 6 admits a solution whose leading term is $2/(3\gamma t)$. To study its stability we make the change of variables $H = v(z)/t$, $t^r = z$. Then this equation takes the form

$$\frac{d}{dz} \left[\frac{1}{2} v^2 + V(v, z) \right] = -\frac{3H_0^r}{r\gamma} v^2 v^{1-r} + O\left(\frac{1}{z^2}\right) \quad (16)$$

$$V(v, z) = \frac{9H_0^r}{2r^2} \left[\frac{v}{4-r} - \frac{2}{3\gamma(3-r)} \right] \frac{v^{3-r}}{z} + O\left(\frac{1}{z^2}\right) \quad (17)$$

$$V(v, z) = \frac{1}{2} H_0^3 \left(v - \frac{2}{3\gamma} \ln v \right) \frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad r = 4 \quad (18)$$

C. Stability of exponential-like solutions

In the case of the solution for $\gamma = 0$ and $m \neq 0$ [8]

$$a(t) = \exp \left[\frac{2m}{2m-1} w_M t^{\frac{2m-1}{2m}} \right] \quad w_M = (3^{m+1} m \alpha)^{-\frac{1}{2m}} \quad (19)$$

we make the change of variables $H = (sw(s)/t)$, $t^{r/(r+1)} = s$ and get

$$\frac{d}{ds} \left[\frac{1}{2} w'^2 + W(w, s) \right] = -\frac{3^{1-r/2}}{\gamma_0} \frac{r+1}{r} \frac{w'^2}{w^{r-1}} s + O\left(\frac{1}{s}\right) \quad (20)$$

where now the potential to leading order in $1/s$ is

$$W(w, s) = -\frac{r+1}{r^2} \left[\frac{3^{1-r/2} w^{3-r}}{(3-r)\gamma_0} + \frac{9(r+1)w^4}{8} \right], \quad r \neq 3 \quad (21)$$

$$W(w, s) = -4 \frac{3^{1-r/2}}{9\gamma_0} \ln w - w^4, \quad r = 3 \quad (22)$$

V. CONCLUSIONS

When $r = 0$, the splitting of the large time asymptotic behavior of solutions in terms of $\text{sgn}(\gamma - \gamma_0)$, closely resembles the classification for the noncausal solutions. However, causality makes new families of solutions appear as the bouncing ones and those which expand from a singularity but recollapse in a finite time at another singularity. Most singular solutions have particle horizons.

We demonstrate that there is no value of r for which there is a stable expanding de Sitter period in the far past. This supports strongly the conclusion of that causality avoids the deflationary behavior proposed by Barrow [8].

If $r < 0$, a stable inflationary phase occurs in the far future for any $\gamma > 0$; the condition $\gamma = \gamma_0$ is required if $r = 0$, and such a behavior is unstable for $r > 0$. We note that the noncausal model has no stable de Sitter solution if $r = 0$. If $\gamma = 0$ and $m < 0$, the same faster than exponential expansion found by Barrow [8] is asymptotically stable.

If $r > 0$, we find that relaxation effects do not alter the perfect fluid behavior $a \sim t^{2/(3\gamma)}$ for $t \rightarrow \infty$. This arises because the viscous pressure decays faster than the thermodynamical pressure. However, if $r = 0$ and $\gamma > \gamma_0$, both pressures decay asymptotically as t^{-2} and the exponent becomes ν_+ . The perfect fluid behavior becomes unstable if $r < 0$.

Large negative viscous pressures may arise in stable evolutions which avoid an initial singularity or have a viscosity-driven inflationary stage. If $r = 0$, no two-parameter solution satisfies the energy conditions.

VI. REFERENCES

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